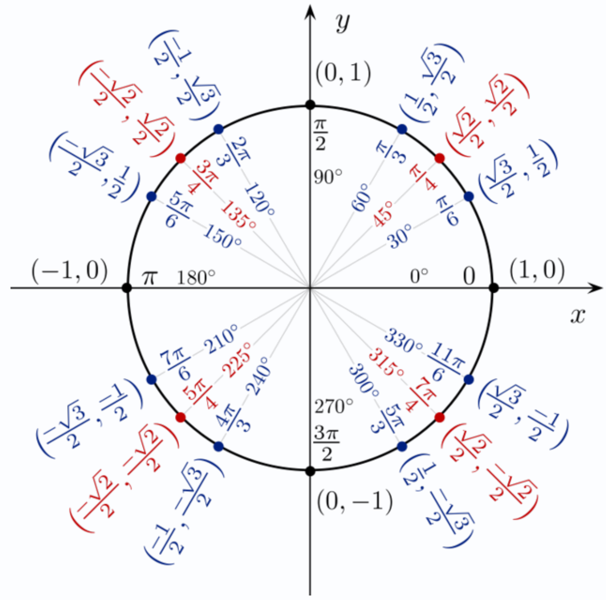
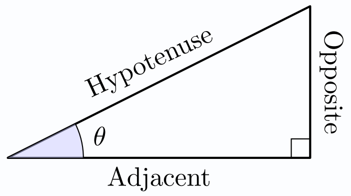
Unit Circle



(Inverse) Trigonometric Functions and Identities



$$\sin\theta = \frac{\mathrm{O}}{\mathrm{H}} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{\mathrm{A}}{\mathrm{H}} = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\tan\theta = \frac{\mathrm{O}}{\mathrm{A}} = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$$

Reciprocal	**Pythagorean**
$\csc\theta = 1/\sin\theta$	$\sin^2\theta + \cos^2\theta = 1$
$\sec\theta = 1/\cos\theta$	$\sec^2\theta - \tan^2\theta = 1$
$\cot\theta = 1/\tan\theta$	$\csc^2\theta - \cot^2\theta = 1$

Even/Odd Identities

$\sin(-\theta) = -\sin\theta$	$\csc(-\theta) = -\csc\theta$
$\cos(-\theta) = \cos\theta$	$\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$	$\cot(-\theta) = -\cot\theta$

Double Angle Identities

$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Angle Sum/Difference and Product-to-Sum

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

Inverse Function	**Domain**	**Range**
$y = \arcsin x = \sin^{-1} x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$y = \arccos x = \cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$y = \arctan x = \tan^{-1} x$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$
$y = \mathrm{arccsc}\, x = \csc^{-1} x$	$\lvert x\rvert \geq 1$	$[-\pi/2,\pi/2]-\{0\}$
$y = \mathrm{arcsec}\, x = \sec^{-1} x$	$\lvert x\rvert \geq 1$	$[0,\pi]-\{\pi/2\}$
$y = \mathrm{arccot}\, x = \cot^{-1} x$	$[-\infty,\infty]$	$(0,\pi)$

Complex Numbers

Cartesian form:	$z = x + iy = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$
Conjugate	$\bar{z} = x - iy$
Polar form:	$z = r(\cos\theta + i\sin\theta) = r\angle\theta$
Modulus	$r = \lvert z\rvert = \sqrt{x^2+y^2}$
$\mathrm{Arg}(z)$	$\theta = \arctan\left(\frac{y}{x}\right),\ \theta\in(-\pi,\pi]$
Exponential form:	$z = re^{i\theta}, \qquad e^{i\theta} = \cos\theta + i\sin\theta$
De Moivre's:	$z^n = r^n\angle(n\theta)$
n^{th} **roots:**	$w_k = r^{\frac{1}{n}}\angle\left(\frac{\theta+2k\pi}{n}\right), k = 0,\cdots,n-1$
$z_1z_2 = r_1r_2\angle(\theta_1+\theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$

(Inverse) Hyperbolic Functions & Identities

$\sinh x = \dfrac{e^x-e^{-x}}{2}$	$\mathrm{csch}\, x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x-e^{-x}}$
$\cosh x = \dfrac{e^x+e^{-x}}{2}$	$\mathrm{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x+e^{-x}}$
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

Square Formulas

$\cosh^2 x - \sinh^2 x = 1$	$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
$\mathrm{sech}^2\, x + \tanh^2 x = 1$	$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$
$\coth^2 x - \mathrm{csch}^2\, x = 1$	

Even/Odd Identities

$\sinh(-x) = -\sinh x$	$\mathrm{csch}(-x) = -\mathrm{csch}\, x$
$\cosh(-x) = \cosh x$	$\mathrm{sech}(-x) = \mathrm{sech}\, x$
$\tanh(-x) = -\tanh x$	$\coth(-x) = -\coth x$

Double Angle Identities

$$\sinh 2x = 2\sinh x\cosh x \qquad \tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$$

Angle Sum/Difference and others

$$\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x\sinh y$$

$$\cosh(x\pm y) = \cosh x\cosh y \pm \sinh x\sinh y$$

$$\tanh(x\pm y) = \frac{\tanh x\pm\tanh y}{1\pm\tanh x\tanh y}$$

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$$

$$\sinh^2 x - \sinh^2 y = \sinh(x+y)\sinh(x-y)$$

Inverse Function	**Domain**	**Range**
$y = \mathrm{arcsinh}\, x = \ln\left(x+\sqrt{x^2+1}\right)$	$(-\infty,\infty)$	$(-\infty,\infty)$
$y = \mathrm{arccosh}\, x = \ln\left(x+\sqrt{x^2-1}\right)$	$[1,\infty)$	$[0,\infty)$
$y = \mathrm{arctanh}\, x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	$(-1,1)$	$(-\infty,\infty)$
$y = \mathrm{arccsch}\, x = \ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}+1}\right)$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$y = \mathrm{arcsech}\, x = \ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}-1}\right)$	$(0,1]$	$[0,\infty)$
$y = \mathrm{arccoth}\, x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$	$(-\infty,1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

Derivatives & Integrals

(Inverse) Trigonometric Functions

$\frac{d}{dx}\sin x = \cos x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + C$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$	
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{x\sqrt{x^2-1}}$	
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$
$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$	

(Inverse) Hyperbolic Functions

$\frac{d}{dx}\sinh x = \cosh x$	$\int \sinh x\,dx = \cosh x + C$
$\frac{d}{dx}\cosh x = \sinh x$	$\int \cosh x\,dx = \sinh x + C$
$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$	$\int \tanh x\,dx = \ln(\cosh x) + C$
$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x\coth x$	$\int \operatorname{csch} x\,dx = \ln\lvert\tanh\frac{x}{2}\rvert + C$
$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x\tanh x$	$\int \operatorname{sech} x\,dx = \tan^{-1}\lvert\sinh x\rvert + C$
$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$	$\int \coth x\,dx = \ln\lvert\sinh x\rvert + C$
	$\int \operatorname{sech}^2 x\,dx = \tanh x + C$
$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{dx}{\sqrt{a^2+x^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C = \ln\lvert x + \sqrt{x^2+a^2}\rvert + C$
$\frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arccosh}\left(\frac{x}{a}\right) + C = \ln\lvert x + \sqrt{x^2-a^2}\rvert + C$
$\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2}$	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert\frac{x+a}{x-a}\right\rvert + C$
$\frac{d}{dx}\operatorname{arccsch} x = \frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
$\frac{d}{dx}\operatorname{arcsech} x = \frac{-1}{x\sqrt{1-x^2}}$	$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{-1}{a}\operatorname{arcsech}\left(\frac{x}{a}\right) + C = \frac{-1}{a}\ln\left\lvert\frac{a+\sqrt{a^2-x^2}}{x}\right\rvert + C$
$\frac{d}{dx}\operatorname{arccoth} x = \frac{1}{1-x^2}$	

Derivatives & Integrals (cont.)

Differentiation	Integration
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	$\int u\,dv = uv - \int v\,du$
$\frac{d}{dx}(uv) = vu' + uv'$	$\int_a^b f(u(x))\frac{du}{dx}dx = \int_{u(a)}^{u(b)} f(u)\,du$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2}$	$\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C$
$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x)$	
$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$	$\int e^{f(x)}\,dx = \frac{e^{f(x)}}{f'(x)} + C$
$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$	$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + C$
$\frac{d}{dx}a^{f(x)} = a^{f(x)}f'(x)\ln a$	

Denominator Factor(s)	Term in partial fractions decomposition
$x-a$	$\frac{A}{x-a}$
$(x-a)^2$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
x^2+ax+b	$\frac{Ax+B}{x^2+ax+b}$

Limit

D.N.E

Unbounded	$f(a) = \frac{b}{0} \neq 0$
Different one-sided limits	$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$
Oscillatory	$\lim_{x\to a}\sin(x)$

Limit Laws & Standard Limits

$+$	$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
$\times$	$\lim_{x\to a}(cf(x)) = c(\lim_{x\to a} f(x)), \quad c\in\mathbb{R}$
	$\lim_{x\to a}(f(x)g(X)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$
$\div$	$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x) \neq 0$
$f(g(x))$	$\lim_{x\to a}(f(g(x))) = f(\lim_{x\to a} g(x))$

$\lim_{x\to\infty}\frac{1}{n^p} = 0 \quad (p>0)$	$\lim_{x\to\infty}\frac{n^p}{a^n} = 0 \quad (a>1, \forall p)$
$\lim_{x\to\infty} r^n = 0 \quad (\lvert r\rvert < 1)$	$\lim_{x\to\infty}\frac{a^n}{n!} = 0 \quad (\forall a)$
$\lim_{x\to\infty} a^{\frac{1}{n}} = 1 \quad (a>0)$	$\lim_{x\to\infty}\frac{ln(n)}{n^p} = 0 \quad (p>0)$
$\lim_{x\to\infty} n^{\frac{1}{n}} = 1$	$\lim_{x\to\infty}\left(1+\frac{a}{n}\right)^n = e^a \quad (\forall a)$
L'Hôpital Rule	$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}$	Fatoring/dividing by the highest power
$\infty\times 0$	Rearranging to $\frac{f(x)}{g(x)}$
$\infty-\infty$	Rearranging into a single fraction
$0^0, \infty^0, 1^\infty$	Taking a logarithm

Integrand Expression	Trigonometric Identity	Trigonometric substitution	Hyperbolic Identity	Hyperbolic substitution
$1-x^2$	$1-\sin^2\theta=\cos^2\theta$	$x=\sin\theta, \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$1-\tanh^2\theta=\operatorname{sech}^2\theta$	$x=\tanh\theta$
$1+x^2$	$1+\tan^2\theta=\sec^2\theta$	$x=\tan\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$1+\sinh^2\theta=\cosh^2\theta$	$x=\sinh\theta$
x^2-1	$\sec^2\theta-1=\tan^2\theta$	$x=\sec\theta, \theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$\cosh^2\theta-1=\sinh^2\theta$	$x=\cosh\theta, \theta\geq 0$

Series Test	Series	Converges if	Diverges if	Comments
Divergence Test	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} a_n\neq 0$	should be the first test used. Inconclusive if limit $=0$.
P-Series Test	$\sum_{n=1}^{\infty}\frac{1}{n^p}$	$p>1$	$p\leq 1$	harmonic series when $p=1$. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n=f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing.
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0\leq a_n\leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0\leq b_n\leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	find a larger series to show convergence, find a smaller series to show divergence.
Alternating Series Test	$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$	$b_n>0 \quad \forall n$, $\lim_{n\to\infty} b_n=0$, $b_{n+1}\leq b_n$	n/a	series must be decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert<1$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert>1$	test fails if $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert=1$.

Polynomial Approximation and Taylor Series

Linear $L(x)\approx f(a)+f'(a)(x-a)$

Quadratic $Q(x)\approx f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2$

Taylor Series $T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

$=Q(x)+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin Series $M(x)=T(x)$ where $a=0$

First Order Differential Equations (DEs)

Separable DEs $\frac{dy}{dx}=F(x)G(y)$

1. Separate x's and y's — $\frac{1}{G(y)}\,dy=F(x)\,dx$
2. Integrate both sides — $\int\frac{1}{G(y)}\,dy=\int F(x)\,dx+C$

Non-trivial solutions $y=h(x)$ — If $|y|=h(x)$, $y=\pm h(x)$. If $y=\pm e^C h(x)$, $y=Ah(x)$

Trivial solutions $y=C$ — Check constant sols $f(C)=0$

Linear DEs $y'+P(x)y=Q(x)$

1. Calculate — $I(x)=e^{\int P(x)\,dx}$
2. Multiply $I(x)$ to DE — $(I(x)y)'=I(x)Q(x)$
3. Integrate both sides — $I(x)y=\int I(x)Q(x)\,dx+C$
4. Divide $I(x)$ both sides — $y=\frac{1}{I(x)}\left(\int I(x)Q(x)\,dx+C\right)$

Second Order Differential Equations (DEs)

Homogeneous Eqs $ay''+by'+cy=0$

1. Write down auxiliary eq — $a\lambda^2+b\lambda+c=0$
2. Solve for λ — $\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
3. Depending on the roots

(i) $b^2-4ac>0$ Two real roots λ_1,λ_2 — $y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}$

(ii) $b^2-4ac=0$ One real root λ — $y=Ae^{\lambda x}+Bxe^{\lambda x}$

(iii) $b^2-4ac<0$ Two complex roots $\alpha\pm i\beta$ — $y=Ae^{\alpha x}\sin\beta x+Be^{\alpha x}\cos\beta x$

Non-homogeneous Eqs $ay''+by'+cy=f(x)$

1. Solve for y_h — $ay''+by'+cy=0$
2. Write down a trial y_p — Check $f(x)$ and $f'(x)$

If terms in trial y_p appear in y_h, include an extra x.

3. Determine derivatives of y_p — y_p', y_p''
4. Subs y_p, y_p', y_p'' into the DE, solve for unknown coeffs.
5. Write down the general sol — $y=y_h+y_p$

Initial value problems (IVP): Conditions specified at the same value, usually the lower boundary of the domain.

Boundary value problems (BVP): Conditions specified at the extremes (boundaries) value of the domain.

Matrix

Identity matrix I_n: A square $n \times n$ matrix, where every entry on the main diagonal is equal to 1.

Transpose matrix A^T: $[A^T]_{m\times n} = [A]_{n\times m}$

E.g. $I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad A^T = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}^T = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$

Gaussian Elimination

1. Set up an augmented matrix $[A|\underset{\sim}{\mathbf{b}}]$
2. Perform elementary row operations:
   (i) Swapping two rows,
   (ii) Multiplying a row by a nonzero number,
   (iii) Adding a multiple of one row to another row
3. Using the above operations until a matrix transformed into an upper triangular matrix or in row echelon form, ie
   (i) The leading entry in each row is to the right of the leading entry in the row above
   (ii) All rows having only zero entries are at the bottom

For example, $A = \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ 0 & a_{22} & a_{23} & a_{24}\\ 0 & 0 & a_{33} & a_{34}\\ 0 & 0 & 0 & 0\end{bmatrix}$

Gauss–Jordan Elimination

4. Using row operations until all the leading entries are 1 and zeros elsewhere (in reduced row echelon form)
5. Solution depends on
   (i) $\text{rank}(A) = \text{rank}([A|\underset{\sim}{\mathbf{b}}]) = n$ Unique solution
   (ii) $\text{rank}(A) \neq \text{rank}([A|\underset{\sim}{\mathbf{b}}])$ No solution
   (iii) $\text{rank}(A) = \text{rank}([A|\underset{\sim}{\mathbf{b}}]) = r < n$ Infinite solutions
   Assign $n-r$ parameter(s) to the non-leading variable(s)

$\text{rank}(A) =$ the number of non-zero rows in row echelon form or the number of linearly independent rows

$\text{trace}(A) =$ the sum of main diagonal entries

Determinants

$\det(A_{2\times 2}) = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = \text{ad-bc}$

$$\det(A_{3\times 3}) = \begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = a\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\begin{vmatrix}d & f\\ h & i\end{vmatrix} + c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$$

Cofactor expansions along i^{th} row or j^{th} column

$$\det(A_{n\times n}) = \sum_{j=1}^{n} a_{ij}C_{ij},\ n > 2$$

$C_{ij} = (-1)^{i+j}M_{ij}$, M_{ij} is the determinant of the submatrix obtained by deleting the i^{th} row and the j^{th} column

1. If A has a row or a column of zeros, $\det(A) = 0$
2. Multiply a row by k to get A', then $\det(A') = k\det(A)$
3. Swap any two rows to get A', then $\det(A') = -\det(A)$
4. Add a multiple of one row to another, $\det(A') = \det(A)$
5. If one row is scalar multiple of another, $\det(A) = 0$
6. $\det(A^T) = \det(A)$
7. $\det(kA) = k^n\det(A)$
8. $\det(AB) = \det(A)\det(B)$
9. $\det(A^{-1}) = 1/\det(A)$
10. If A is a triangular matrix, $\det(A) = \prod_{i=1}^{n} a_{ii}$

Matrix inverses

If $\det(A) = 0$ not invertible & singular

If $\det(A) \neq 0$ invertible & non-singular such that: $A\,A^{-1} = I$ and $A^{-1}\,A = I$

$$A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \implies A^{-1} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

For $A_{n\times n}$, compute A^{-1} using **Gaussian elimination**:

1. Set up an augmented matrix $[A|I]$
2. Perform row operations to convert the LHS into I, which causes the RHS to become A^{-1}, ie $[I|A^{-1}]$

Methods for solving systems of linear eqs

For systems of n eqs w/ n unknowns: $A\underset{\sim}{\mathbf{x}} = \underset{\sim}{\mathbf{b}}$, $\det(A) \neq 0$

I. **Using inverse** $\underset{\sim}{\mathbf{x}} = A^{-1}\underset{\sim}{\mathbf{b}}$

II. **Cramer's rule** $x_i = \dfrac{\det(A_i)}{\det(A)}$, $i = 1, 2, ..., n$
(Replace i^{th} column with $\underset{\sim}{\mathbf{b}}$)

Equations of lines and planes in space

⋆ A **line** passing through pts $P(x_0, y_0, z_0)$, $Q(x, y, z)$

Direction vector $\overrightarrow{PQ} = [x, y, z] - [x_0, y_0, z_0] = t\underset{\sim}{\boldsymbol{v}}$,
For $\underset{\sim}{\boldsymbol{v}} = [a, b, c]$, $[x, y, z] = [x_0, y_0, z_0] - t[a, b, c]$

Vector equation $\underset{\sim}{\boldsymbol{r}} = \underset{\sim}{\boldsymbol{r_0}} + t\ \underset{\sim}{\boldsymbol{v}}$

Parametric equations $\begin{cases}x = x_0 + ta\\ y = y_0 + tb\\ z = z_0 + tc\end{cases}$

Cartesian equation $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

⋆ A **plane** passing through pts P, Q and R

Direction vectors $\underset{\sim}{\boldsymbol{u}} = \overrightarrow{PQ}$, $\underset{\sim}{\boldsymbol{v}} = \overrightarrow{PR}$
any two direction vectors will do

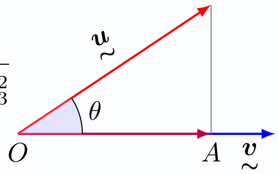
Vector equation $\underset{\sim}{\boldsymbol{r}} = \underset{\sim}{\boldsymbol{r_0}} + t\underset{\sim}{\boldsymbol{u}} + s\underset{\sim}{\boldsymbol{v}}$, $t, s \in \mathbb{R}$

Cartesian equation $\underset{\sim}{\boldsymbol{n}} \cdot \underset{\sim}{\boldsymbol{r}} = \underset{\sim}{\boldsymbol{n}} \cdot \overrightarrow{OP}$, where $\underset{\sim}{\boldsymbol{n}} = \underset{\sim}{\boldsymbol{u}} \times \underset{\sim}{\boldsymbol{v}}$

Distance btw a plane and a point $d = \left|\dfrac{\overrightarrow{QP} \cdot \underset{\sim}{\boldsymbol{n}}}{||\underset{\sim}{\boldsymbol{n}}||}\right|$

Vector

$\underset{\sim}{\boldsymbol{u}} = [u_1, u_2, u_3]$, $\underset{\sim}{\boldsymbol{v}} = [v_1, v_2, v_3]$



Vector norms $||\underset{\sim}{\boldsymbol{u}}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Unit vector $\hat{\underset{\sim}{\boldsymbol{u}}} = \underset{\sim}{\boldsymbol{u}}\,/\,||\underset{\sim}{\boldsymbol{u}}||$

Dot product $\underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{v}} = u_1v_1 + u_2v_2 + u_3v_3 = ||\underset{\sim}{\boldsymbol{u}}||\,||\underset{\sim}{\boldsymbol{v}}||\cos\theta$

Angle $\angle(\underset{\sim}{\boldsymbol{u}}, \underset{\sim}{\boldsymbol{v}})$ $\cos\theta = (\underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{v}})\,/\,(||\underset{\sim}{\boldsymbol{u}}||\,||\underset{\sim}{\boldsymbol{v}}||)$

Scalar projection of $\underset{\sim}{\boldsymbol{u}}$ onto $\underset{\sim}{\boldsymbol{v}}$ (length of $\overrightarrow{OA}$) $p = \underset{\sim}{\boldsymbol{u}} \cdot \hat{\underset{\sim}{\boldsymbol{v}}} = ||\underset{\sim}{\boldsymbol{u}}||\cos\theta$

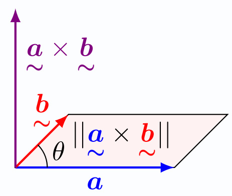
Vector projection of $\underset{\sim}{\boldsymbol{u}}$ onto $\underset{\sim}{\boldsymbol{v}}$ (vector $\overrightarrow{OA}$) $\underset{\sim}{\boldsymbol{p}} = p\,\hat{\underset{\sim}{\boldsymbol{v}}} = (\underset{\sim}{\boldsymbol{u}} \cdot \hat{\underset{\sim}{\boldsymbol{v}}})\hat{\underset{\sim}{\boldsymbol{v}}}$

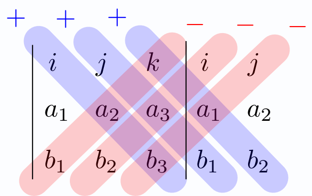
Cross product in $\mathbb{R}^3$ $\underset{\sim}{\boldsymbol{a}} = [a_1, a_2, a_3]$, $\underset{\sim}{\boldsymbol{b}} = [b_1, b_2, b_3]$

$\underset{\sim}{\boldsymbol{n}} = \underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} =$ normal vector perpendicular to both $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$

If $\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} = \underset{\sim}{\mathbf{0}}$, $\underset{\sim}{\boldsymbol{a}} \parallel \underset{\sim}{\boldsymbol{b}}$. $||\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}|| = ||\underset{\sim}{\boldsymbol{a}}||\,||\underset{\sim}{\boldsymbol{b}}||\sin\theta$

$$\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} = \begin{vmatrix}\underset{\sim}{\boldsymbol{i}} & \underset{\sim}{\boldsymbol{j}} & \underset{\sim}{\boldsymbol{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \underset{\sim}{\boldsymbol{i}}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \underset{\sim}{\boldsymbol{j}}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \underset{\sim}{\boldsymbol{k}}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$
$$= [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$$





Scalar triple product: the volume of parallelepiped

$$\underset{\sim}{\boldsymbol{a}} \cdot (\underset{\sim}{\boldsymbol{b}} \times \underset{\sim}{\boldsymbol{c}}) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$