SUPPLY CHAIN MODELLING AND OPTIMIZATION

TOPIC 1 INTRODUCTION

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WeChat group: YSU Economics & Supply Chain

ASSESSMENTS

1. **20% Assignment** (23 June 11:59 PM)
   - Submit to WeChat ID: **cubdwi**
   - Tell me your name (in English) and student ID, and wait for confirmation.
2. **80% Final Exam** (Duration: 2 hours)

CORE READING

Ghiani, Gianpaolo, Gilbert Laporte, and Roberto Musmanno, 2003. "**Introduction to logistics systems planning and control**". https://doi.org/10.1002/0470014040.

COURSE CALENDAR

Date	Topic	Topic
Sun 15	1	Introduction + Review of Optimisation problems
	2	Network Design - SESC
Mon 16	2	Network Design - TEMC
Tue 17	4	Network Design under Uncertainty (CF analysis & Decision tree)
Wed 18	5	Forecasting Techniques (LR, MA, ES, Holt-Winter Model)
Thu 19	6	Inventory Model: Single-Commodity
Fri 20	7	Inventory Model: Discount and Multi-Commodity
Sat 21	8	Stochastic Inventory Model: Single-Period Model
Sun 22	9	Stochastic Inventory Model: Multi-Period Model
Mon 23		Revision + Assignment due at 11:59 PM

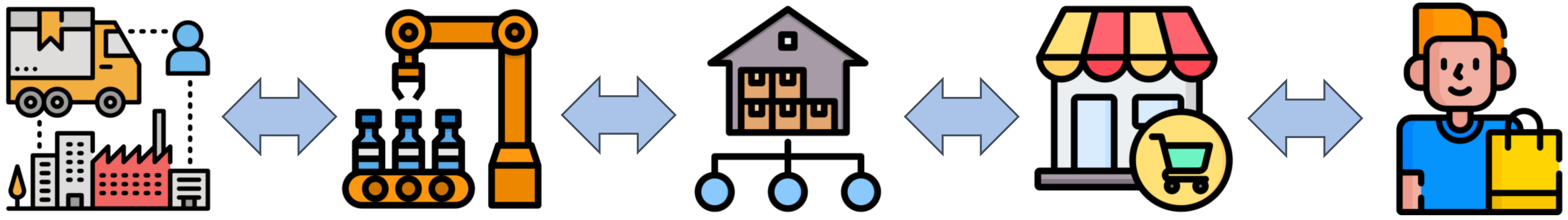
TOPIC 1: INTRO & REVIEW OF OPTIMISATION

1. Introduction to supply chain
   1.1. What is supply chain?
   1.2. Supply chain network flows
   1.3. Roles of logistics in supply chain
   1.4. Supply chain management (SCM)
   1.5. Supply chain network design
   1.6. Static facilities in supply chain

2. Review of optimisation problems
   2.1. Unconstrained optimisation
   2.2. Constrained optimisation

1.1 WHAT IS SUPPLY CHAIN?

A network of individuals and companies involved in the **production** and **distribution** of goods and services.



Suppliers

Providing raw materials to manufacturers

Manufacturers

Producing goods in large numbers

Distributors

Distributing goods to shops and companies

Resellers
Warehouses

Retailers

Selling goods to the public

Supermarkets
Shops

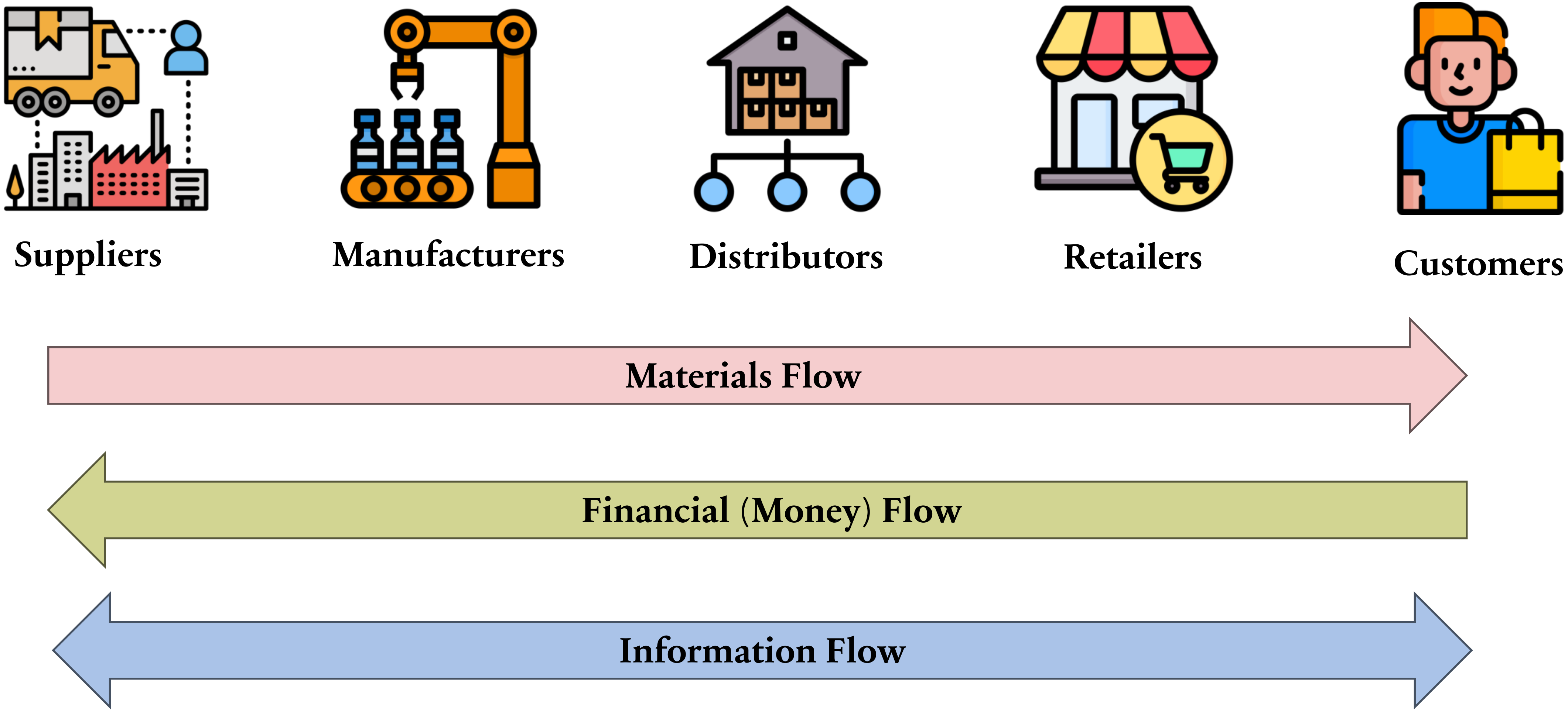
Customers

Buying goods or service

Logistics

the process of coordinating the efficient flow of goods, services, and related information. It involves planning, implementing, and controlling the movement and sometimes warehousing of goods throughout the supply chain. Logistics is a critical component that holds the supply chain together.

1.2 SUPPLY CHAIN NETWORK FLOWS



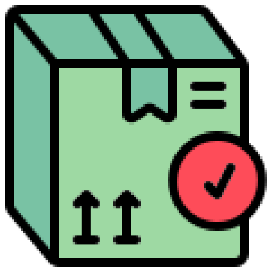
1.3 ROLES OF LOGISTICS IN SUPPLY CHAIN

Transportation management

Planning, organising, and executing movement of goods from one place to another.

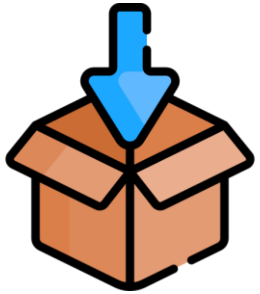
Warehousing & inventory management

Managing inventory levels, storing, and retrieving products



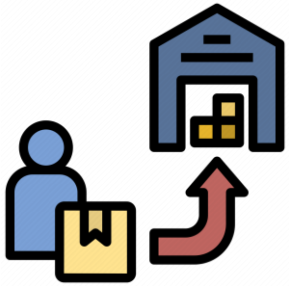
Order fulfillment

Handling customer orders from initial receipt to delivery. (Customer services)



Packaging

Designing, producing, and delivering packaging materials and supplies to warehouses



Reverse logistics

Returning goods to their original manufacturer after they have been used.

1.4 SUPPLY CHAIN MANAGEMENT (SCM)

A process of **integrating, planning, sourcing, making and delivering products, from raw materials to end customers,** and measuring the results globally to satisfy customers and make a profit. SCM includes

- Demand planning
- Supplier relationship management
- Risk management
- Strategic sourcing
- Collaboration
- Performance metrics

Lead time (n.) = the amount of time it takes for a product to move from the start until its conclusion

1.5 SUPPLY CHAIN NETWORK DESIGN

Supply chain network design is the strategic configuration of a supply chain with the following **objectives**

- minimising transportation costs
- reducing lead times
- improving customer service
- minimising inventory levels
- maximising overall SC profitability.

Factors influencing network design includes:

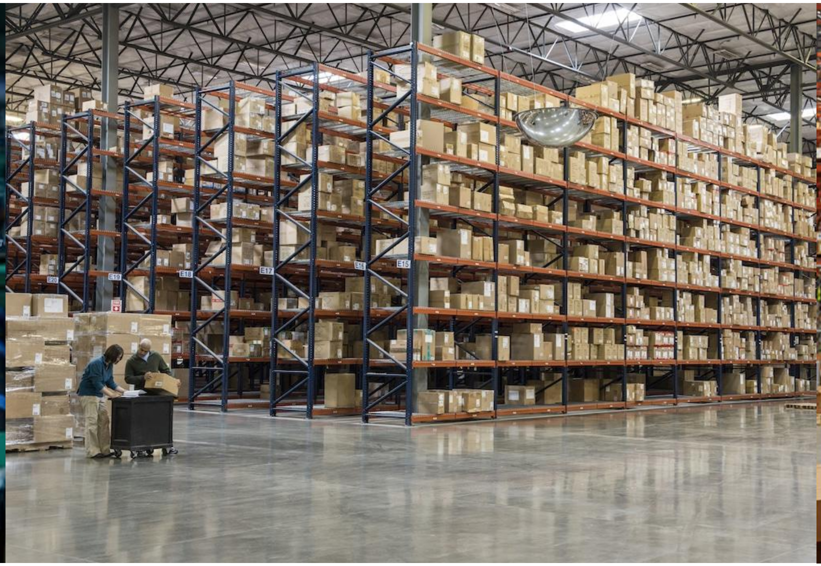
- Demand Forecasting
- Location of Facilities
- Inventory Management
- Network Capacity
- Supplier & Vendor
- Management
- Customer Service Levels
- Technology & Information Systems
- Regulatory & Compliance
- Cost Analysis
- Risk Management
- Sustainability & Environmental Impact
- Competitive Factors
- Market Dynamics
- Globalisation

1.6 STATIC FACILITIES IN SUPPLY CHAIN

Facility	Roles in the SC	Operations
Plant/Factory	Supplier/manufacturer	Produces or manufactures goods
Warehouse	Distributor	Stores & holds inventory, often receives shipments from suppliers
Distribution centre	Distributor/Wholesaler	Receives, processes, and sends out orders to customers
Fulfillment centre	Distributor	Reorders and prepares products for shipping to customers
Cross dock	Distributor	Direct transfer of goods from vehicles to vehicles
Locker	Distributor	Stores non-perishable materials temporarily while being moved between facilities (e.g., inventory, shipping).
Service outlet	Retailer/Service provider	Provides value-added services to customers (repair, maintenance, consulting).
Parking lot	Dispatcher	Supports the movement of personnel and vehicles within the facility.
Collection centre	Collector	Collects & stores waste materials/hazardous goods temporarily until disposal.
Disposal centre	Disposer	Handles the removal and destruction of unwanted or hazardous materials.

Plant/Factory

Warehouse

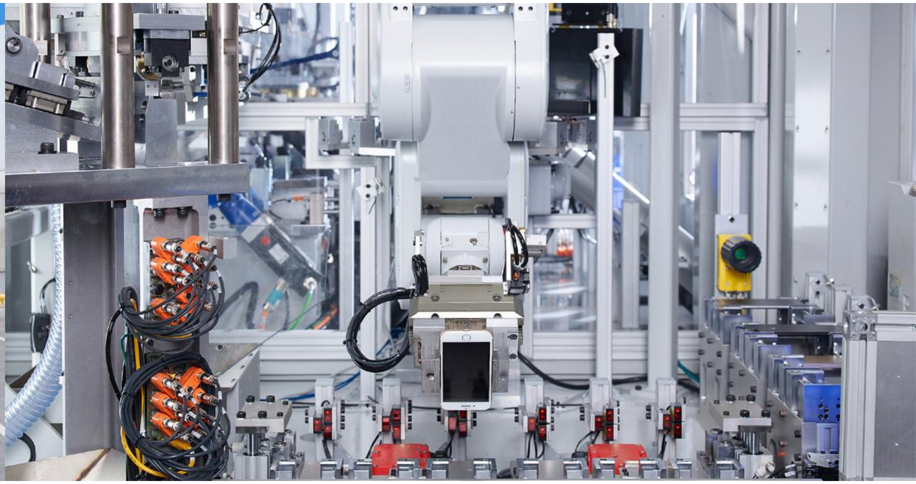


Distribution centre

Fulfillment centre & service outlet

Cross dock

Disposal/Recycle centre



2 REVIEW OF OPTIMISATION PROBLEMS

Optimisation is the mathematical discipline that deals with **maximising** and **minimising** functions under certain conditions or constraints.

These functions are called **objective functions.**

There are 2 types of optimisation problems, depending on whether there is **constraint** on the **decision variables** x and y.

Unconstrained optimisation

$$\min f(x,y) = x^2 + 2y^2$$

Constrained optimisation

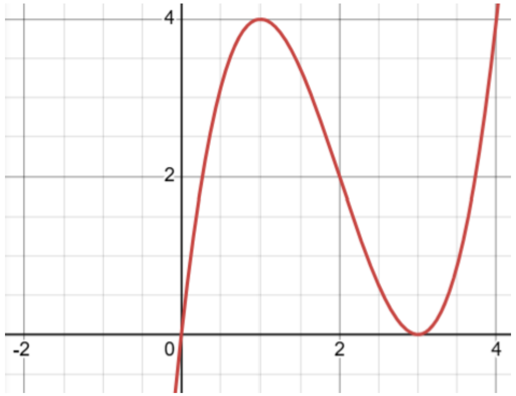
$$\min f(x,y) = x^2 + 2y^2$$

$$-2 < x < 5, \quad y \geq 1$$

2.1.1 LOCAL MAXIMUM AND MINIMUM

1st order	2nd order	
$f'(x^*) = 0$, x^* is the critical point	$f''(x^*) < 0$,	x^* is (local) max.
	$f''(x^*) > 0$,	x^* is (local) min.
	$f''(x^*) = 0$,	x^* can be max, min or neither.

Example. Find the local maximum and minimum points of the function $f(x) = x^3 - 6x^2 + 9x$



$$\begin{aligned} f'(x) = \ 3x^2 - 12x + 9 &= 0 \\ 3\left(x^2 - 4x + 3\right) &= 0 \\ 3(x-3)(x-1) &= 0 \\ x^* &= 1,3 \end{aligned}$$

$f(1) = 4 \rightarrow (\mathbf{1}, \mathbf{3})$ and $f(3) = 0 \rightarrow (\mathbf{3}, \mathbf{0})$

$f''(x) = \ 6x - 12 = 6(x-2)$

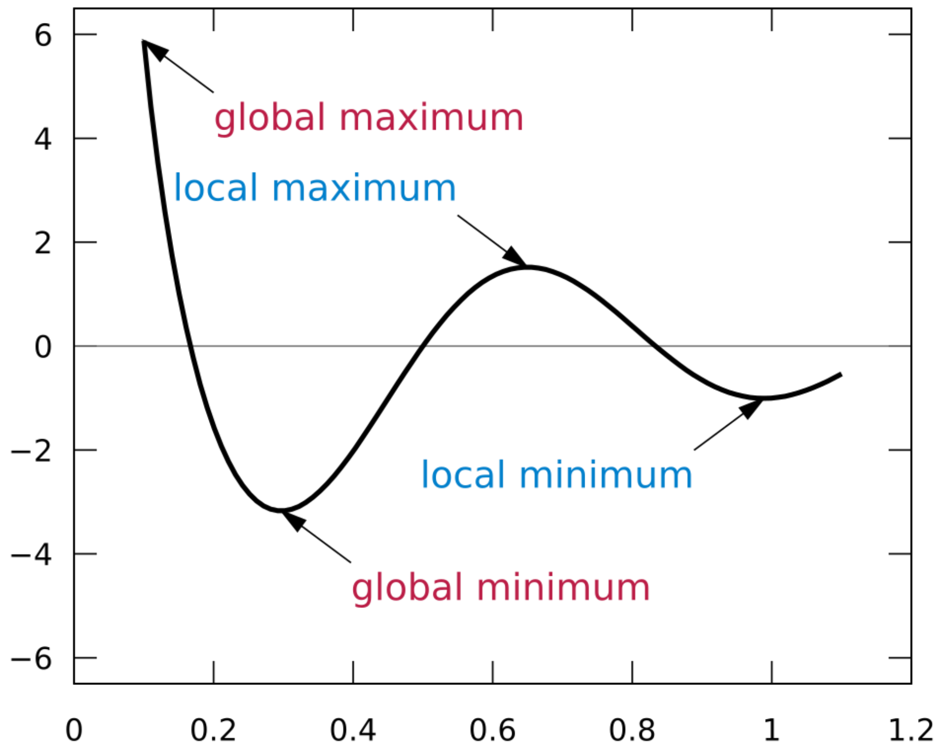
$f''(1) = -6 \quad < 0 \ \rightarrow$ **local max at** $(\mathbf{1}, \mathbf{4})$

$f''(3) = 6 \quad > 0 \ \rightarrow$ **local min at** $(\mathbf{3}, \mathbf{0})$

Are these the ultimate maximum and minimum of the function $f(x)$?

2.1.2 GLOBAL MAXIMUM AND MINIMUM

Optima can occur in the three places:



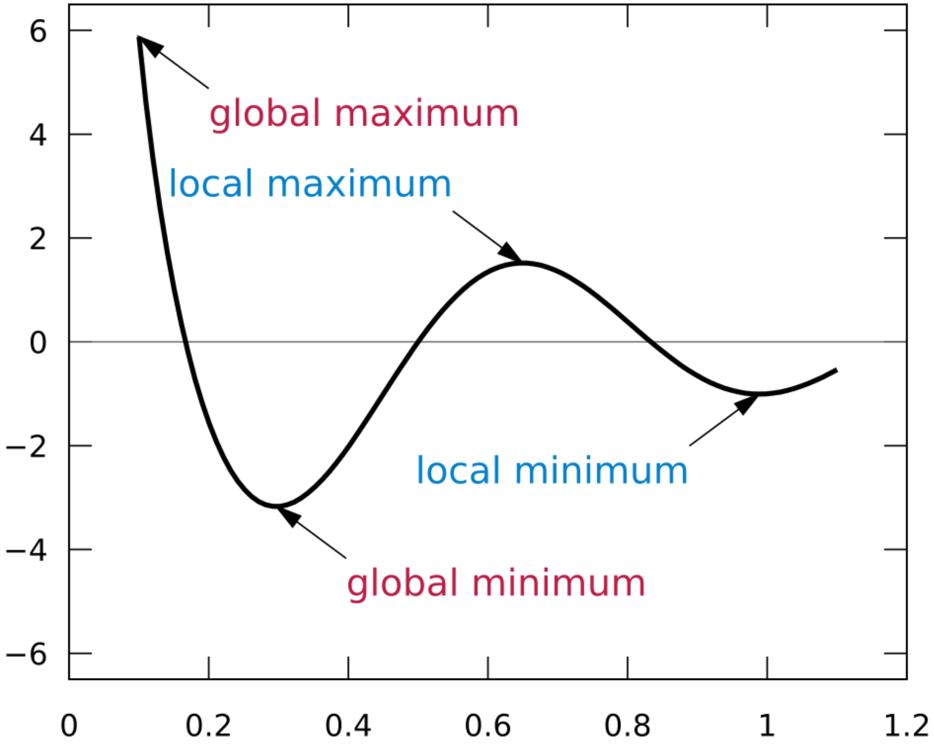
1. The boundary of the domain
2. A non-differentiable point
3. A point x^* with $\nabla f(x^*) = 0$. (i.e. $f'(x^*) = 0$ for 1D)

Then,

- A **local maximum** is a point that $f(x^*) \geq f(x)\ \forall x$ in <u>some open interval containing x^*.</u>
- A **local minimum** is a point that $f(x^*) \leq f(x)\ \forall x$ in <u>some open interval containing x^*.</u>

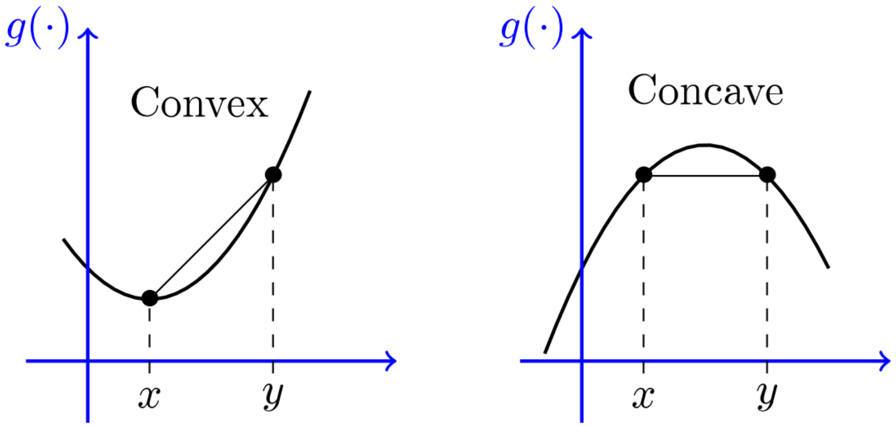
- A **global maximum** is a point that $f(x^*) \geq f(x)\ \forall x$ in <u>the domain of f.</u>
- A **global minimum** is a point that $f(x^*) \leq f(x)\ \forall x$ in <u>the domain of f.</u>

2.1.2 GLOBAL MAXIMUM AND MINIMUM



From the second derivatives, we can conclude that

- When $f''(x) \geq 0\ \forall x$, i.e., $f(x)$ is a **convex** function, then the local minimum x^* is the **global minimum** of $f(x)$.
- When $f''(x) \leq 0\ \forall x$, i.e., $f(x)$ is a **concave** function, then the local maximum $x*$is the **global maximum** of $f(x)$.



2.1.2 GLOBAL MAXIMUM AND MINIMUM

Example. Find the local and global maximum and minimum points of the function

$$f(x) = x^3 + 3x^2 - 2x + 1 \text{ for } x \in [-4,2]$$

$$f'(x) = 3x^2 + 6x - 2 = 0$$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-6 \pm \sqrt{6^2 - 4(3)(-2)}}{2(3)}$$

$$x = -1 - \sqrt{15}/3, \quad -1 + \sqrt{15}/3$$

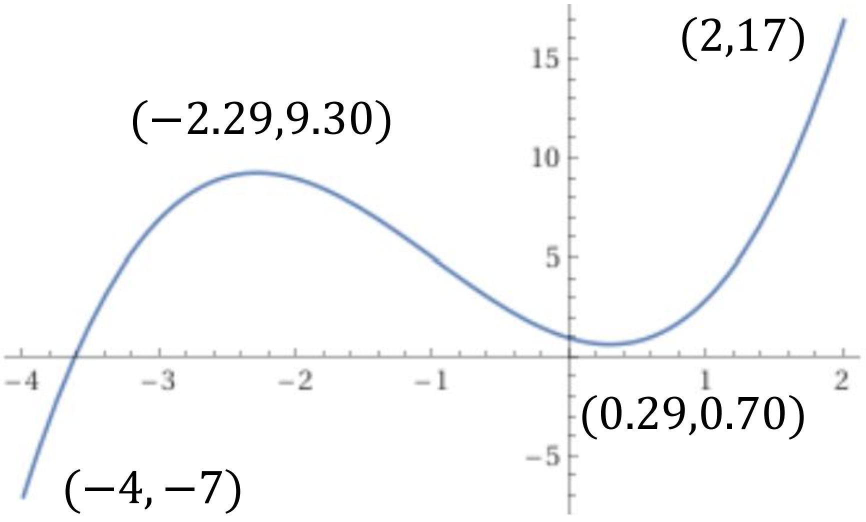
$(-2.29, 9.30)$ and $(0.29, 0.70)$

$$f''(x) = 6x + 6$$

Check the boundaries:

$f(-4) = -7 \ \rightarrow (-4, -7)$ **is global min**

$f(2) = 17 \ \rightarrow (2, 17)$ **is global max**



$f''\left(-1 - \sqrt{15}/3\right) = -7.75 < 0 \ \rightarrow (-2.29,9.30)$ **is (local) max**

$f''\left(-1 + \sqrt{15}/3\right) = 7.75 > 0 \ \rightarrow (0.29,0.70)$ **is (local) min**

2.1.3 HESSIAN MATRIX (2ND DERIVATIVE TEST)

To find the maximum and minimum values for a function $f(x)$ of several independent variable x, we

1. calculate $\nabla f(x)$ and set it to zero,
2. solve the equation to get a solution vector x^*,
3. calculate $\nabla^2 f(x)$, and
4. evaluate $\nabla^2 f(x)$ at x^*.

For a scalar function of several variables $f : \mathbb{R}^n \to \mathbb{R}$, we conduct second derivative test by setting up the **Hessian matrix** given by

$$\mathbf{H}(\mathbf{x}) \equiv \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

2.1.3 HESSIAN MATRIX (2ND DERIVATIVE TEST)

Quadratic Form Test (Definition-Based)

Suppose that x^* is a critical point of $f(x)$, i.e., $\nabla f(\mathbf{x}^*) = 0$, and $\mathbf{H}(\mathbf{x}^*)$ is symmetric matrix. The expression

$$Q(\mathbf{x}) = \mathbf{x}^T\mathbf{H}\mathbf{x}$$

is a **scalar** known as the quadratic form associated with $\mathbf{H}$. Then, we can conclude that

$Q(\mathbf{x})$	**Hessian Type**	**Critical Point Conclusion**
> 0 for all $\mathbf{x} \neq 0$	Positive definite	Local minimum
< 0 for all $\mathbf{x} \neq 0$	Negative definite	Local maximum
≥ 0, some zero	Positive semi-definite	Inconclusive – min or saddle
≤ 0, some zero	Negative semi-definite	Inconclusive – max or saddle
$= 0$ for all $\mathbf{x} \neq 0$	Zero matrix	Inconclusive – higher-order needed
$\exists \mathbf{x} : Q > 0, \exists \mathbf{y}: Q < 0$	Indefinite	Saddle point

2.1.3 HESSIAN MATRIX (2^{ND} DERIVATIVE TEST)

Eigenvalue Test

For any $n \times n$ square matrix $\mathbf{H}$, the eigenvalues λ are the roots of the **characteristic polynomial:**

$$\det(\mathbf{H} - \lambda \boldsymbol{I}) = 0$$

This gives a degree-n polynomial in λ. The roots of this polynomial are the eigenvalues.

Eigenvalues of H	Hessian Type	Critical Point Conclusion
All positive	Positive definite	Local minimum
All negative	Negative definite	Local maximum
All positive, at least one zero	Positive semi-definite	Inconclusive – min or saddle
All negative, at least one zero	Negative semi-definite	Inconclusive – max or saddle
All zero	Zero matrix	Inconclusive – higher-order needed
Mixed signs (some +, some -)	Indefinite	Saddle point

2.1.3 HESSIAN MATRIX (2ND DERIVATIVE TEST)

Eigenvalue (Review)

For any $n \times n$ square matrix $\mathbf{A}$, the eigenvalues λ are the roots of the **characteristic polynomial:**

$$\det(\mathbf{A} - \lambda \boldsymbol{I}) = 0$$

For a symmetric 2×2 matrix, $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(\mathbf{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (\mathrm{a} - \lambda)(d - \lambda) - bc$$

For a symmetric 3×3 matrix, $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\det(\mathbf{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{vmatrix}$$

$$= (a - \lambda)\begin{vmatrix} e - \lambda & f \\ h & i - \lambda \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i - \lambda \end{vmatrix} + \mathrm{c}\begin{vmatrix} d & e - \lambda \\ g & h \end{vmatrix}$$

For matrix size $n \times n, n > 4$, there's no general formula and numerical methods are needed for calculation.

2.1.3 HESSIAN MATRIX (2ND DERIVATIVE TEST)

Example. Find to local maximum and minimum points of $f(x, y) = x^3 - y^3 + 9xy$.

1. **Calculate the first order partial derivatives, i.e., gradient of $\boldsymbol{f(x, y)}$ and set them equal to zero**

$$\nabla f(x, y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 + 9y \\ -3y^2 + 9x \end{pmatrix} = 0$$

2. **Solve for $(\boldsymbol{x^*, y^*})$**

From $\partial f / \partial x$, $\quad 3x^2 + 9y = 0 \ \rightarrow \ y = -\dfrac{x^2}{3}$

Substitute into $\partial f / \partial y$, $\ -3\left(-\dfrac{x^2}{3}\right)^2 + 9x = x\left(-\dfrac{x^3}{3} + 9\right) = 0 \qquad \rightarrow x = 0, 3$

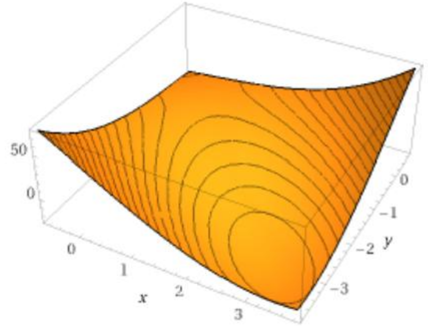
Critical points are $\quad (0, 0)$ and $(3, -3)$

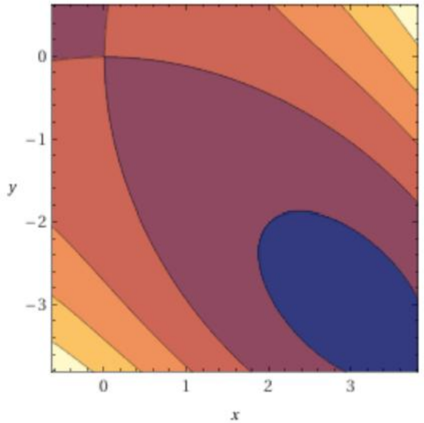
2.1.3 HESSIAN MATRIX (2ND DERIVATIVE TEST)

Example. Find the local maximum and minimum points of $f(x, y) = x^3 - y^3 + 9xy$.

3. Compute the Hessian matric of $f(x, y)$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 9 \\ 9 & -6y \end{bmatrix}$$





4. Evaluate the Hessian at critical points

At $(0, 0)$: $H = \begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix} \rightarrow \det(H - \lambda I) = \begin{vmatrix} -\lambda & 9 \\ 9 & -\lambda \end{vmatrix} = \lambda^2 - 9^2 = (\lambda - 9)(\lambda + 9) = 0 \quad \rightarrow \lambda = -9, 9$

∴ Mixed signs → Indefinite → Saddle point at $(0, 0)$

At $(3, \text{-}3)$: $H = \begin{bmatrix} 18 & 9 \\ 9 & 18 \end{bmatrix} \rightarrow \det(H - \lambda I) = \begin{vmatrix} 18-\lambda & 9 \\ 9 & 18-\lambda \end{vmatrix} = (18-\lambda)^2 - 9^2 = (9 - \lambda)(27 - \lambda) = 0$

$\rightarrow \lambda = 9, 27$

∴ Both positive → Positive definite → Local minimum at $(3, \text{-}3)$

2.1.3 HESSIAN MATRIX IN ACTION

Example (Discriminating Monopolist)

A monopolist producing a single product has two types of customers.

- If it produces q_1 units for type 1 customers, then these customers are willing to pay a price of $50 - 5q_1$ per unit.
- If it produces q_2 units for type 2 customers, then these customers are willing to pay a price of $100 - 10q_2$ per unit.
- The monopolist's cost of manufacturing q units of output (where $q = q_1 + q_2$) is given by $90 + 20q$.

In order to **maximise profit**, how much should the monopolist produce for each market?

0. Write the profit function

The profit is given by $\boxed{f(q_1, q_2) = q_1(50 - 5q_1) + q_2(100 - 10q_2) - (90 + 20(q_1 + q_2))}$

2.1.3 HESSIAN MATRIX IN ACTION

The profit is given by $\boxed{f(q_1, q_2) = q_1(50 - 5q_1) + q_2(100 - 10q_2) - (90 + 20(q_1 + q_2))}$

1. **Find the critical points** $\nabla f(q_1, q_2) = \begin{bmatrix} 50 - 10q_1 - 20 \\ 100 - 20q_2 - 20 \end{bmatrix} = 0$

$$q_1 = 3 \qquad \text{and} \qquad q_2 = 4$$

2. **Setup Hessian matrix** $\nabla^2 f(q_1, q_2) = \begin{bmatrix} -10 & 0 \\ 0 & -20 \end{bmatrix}$

3. **Determine the eigenvalue(s) of the Hessian matrix ~~at the critical points~~**

$$\rightarrow \det(H - \lambda I) = \begin{vmatrix} -10 - \lambda & 0 \\ 0 & -20 - \lambda \end{vmatrix} = (\lambda + 10)(\lambda + 20) = 0 \qquad \rightarrow \lambda = -10, -20$$

$\therefore$ Both negative $\rightarrow$ Negative definite $\rightarrow$ Local maximum at $(3, 4)$ output level

2.1.4 EXERCISES

Use the following functions to practice finding and classifying critical points. For each function:

- Find all critical points by setting the partial derivatives to zero.
- Form the Hessian matrix at each critical point.
- Classify each critical point as a local minimum, local maximum, saddle point, or inconclusive using the second derivative test.

1. $f(x, y) = (x - 2)^2 + 2(y - 1)^2$
2. $f(x, y) = x^3 + y^3 - 3xy$
3. $f(x, y) = 2xy - x^4 - x^2 - y^2$
4. $f(x, y) = e^{-(x^2+y^2)}$
5. $f(x, y) = \sin(x)\sin(y)$
6. $f(x, y) = \ln(1 + x^2 + y^2)$
7. $f(x, y, z) = x^2 + y^2 - z^2$

2.2 CONSTRAINED OPTIMIZATION (COP)

Lagrange Multiplier Method to Solve COP

$$\begin{array}{ll} \text{max/min} & f(x_1, x_2, \dots, x_n) \\ \text{subject to} & g_i(x_1, x_2, \dots, x_n) = 0, \qquad i = 1,2, \dots, m \end{array}$$

1. **Construct the Lagrange function** $F = f(x_1, x_2, \dots, x_n) - \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \dots, x_n) = f - \lambda g$
2. **Find all values of $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ such that**

$$\begin{cases} \dfrac{\partial F}{\partial x_i} = 0, & (i = 1,2, \dots, n) \\ g_i = 0, & (i = 1,2, \dots, m) \qquad \left(\text{obtained from } \dfrac{\partial F}{\partial \lambda_i} = 0\right) \end{cases}$$

3. **Evaluate $\boldsymbol{f}(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ at all the points that result from step 2.**
   - The largest of these values is the max of f subject to the constraints;
   - The smallest is the min of f subject to the constraints.

2.2 CONSTRAINED OPTIMIZATION (COP)

Example. Find the extreme values of $f(x,y) = x^2 - 2y^2$ on the circle $x^2 + y^2 = 1$

In this case, our constraint is $g(x,y) = x^2 + y^2 - 1 = 0$

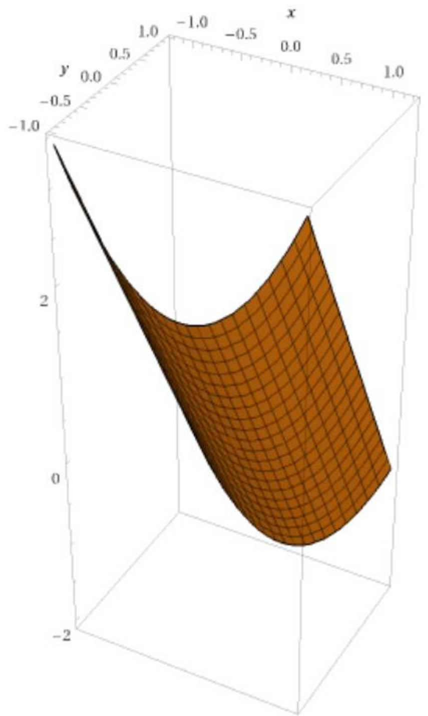
1. **Construct the Lagrange function**

$$F = f - \lambda g = x^2 - 2y^2 - \lambda(x^2 + y^2 - 1)$$

2. **Find all values of $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$**

The condition for extreme values are
$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x - 2\lambda x = 0, \\ \dfrac{\partial F}{\partial y} = 4y - 2\lambda y = 0, \\ \dfrac{\partial F}{\partial \lambda_i} = -(x^2 + y^2 - 1) = 0 \end{cases}$$

2.2 CONSTRAINED OPTIMIZATION (COP)

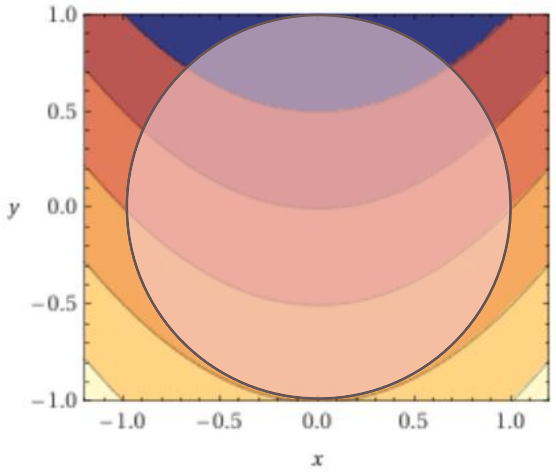


$$\begin{cases} \quad \dfrac{\partial F}{\partial x} = 2x - 2\lambda x = 0, & \rightarrow \; x(1-\lambda) = 0 \quad (1) \\ \quad \dfrac{\partial F}{\partial y} = 4y - 2\lambda y = 0, & \rightarrow \; y(2-\lambda) = 0 \quad (2) \\ \dfrac{\partial F}{\partial \lambda_i} = -(x^2 + y^2 - 1) = 0 \quad (3) \end{cases}$$

From (1), $x = 0$ or $\lambda = 1$.

- If $x = 0$, (3) yields $y = 1, \; -1$
- If $\lambda = 1$, (2) yields $y = 0$, and consequently (3) yields $x = 1, -1$.

Hence, f has possible extreme values at the points (0, 1), (0, -1), (1, 0), (-1, 0)



3. Evaluating f at these points gives

$$f(0,1) = 2, \qquad f(0,-1) = 2, \qquad f(1,0) = 1, \qquad f(-1,0) = 1$$

$$\therefore \min f = f(\pm 1, 0) = 1, \qquad \max f = f(0, \pm 1) = 2$$

2.2 EXERCISES

Use the method of Lagrange multipliers to find and classify the extrema (local or global) for the following problems:

1. $f(x, y) = x^2 + y^2$ subject to $x + y = 1$
2. $f(x, y) = xy$ subject to $x^2 + y^2 = 1$
3. $f(x, y) = x^2 + 4xy + y^2$ subject to $x + y = 3$
4. $f(x, y) = e^x y$ subject to $x^2 + y^2 = 1$
5. $f(x, y) = x^3 y$ subject to $x^2 + y^2 = 1$
6. $f(x, y) = \sin(x) + \cos(y)$ subject to $x + y = \pi/2$
7. $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$

2.2.1 MIXED CONSTRAINED OPTIMIZATION

A general mixed constrained multi-dimensional problem is

Minimise	$f(x), \quad x \in R^n$		
subject to	$h_i(x) = 0,$	$i = 1,2, \dots, m$	Equality constraints
	$g_j(x) \le 0,$	$j = 1,2, \dots, p$	Inequality constraints

The Lagrangian function is

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x),$$

where λ_i are the multipliers for the equality constraints and μ_j for the inequality constraints.

The **Karush-Kuhn-Tucker (KKT) conditions** are necessary for a point x^* to be a local minimum (assuming suitable regularity conditions).

2.2.1 MIXED CONSTRAINED OPTIMIZATION

The Karush-Kuhn-Tucker (KKT) conditions

1. **Stationarity**

   Compute the first-order partial derivatives of L with respect to x_i, λ and μ, and set them equal to zero. Solve for x_i λ and μ

   For minimising $f(x)$: $$\nabla_x L(x^*, \lambda^*, \mu^*) = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*) = 0$$

   For maximising $f(x)$: $$\nabla_x L(x^*, \lambda^*, \mu^*) = -\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*) = 0$$

2. **Primal feasibility** $h(x^*) = 0, \quad i = 1, \ldots, m,$ and $g_j(x^*) \leq 0, \quad j = 1, \ldots, p.$

3. **Dual feasibility** $\mu_j^* \geq 0, \quad j = 1, \ldots, p$

4. **Complementary slackness** $\mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, p$

2.2.1 MIXED CONSTRAINED OPTIMIZATION

Example. Minimize $x^2 + y^2$

subject to $h(x, y) = x + y - 1 = 0$

$g(x, y) = x - 0.5 \le 0.$

1. **Form the Lagrangian** $L = x^2 - y^2 + \lambda(x + y - 1) + \mu(x - 0.5)$

2. **Compute the partial derivatives (stationarity)**

$$\nabla_x L = 2x + \lambda + \mu = 0, \quad (1)$$

$$\nabla_y L = 2y + \lambda = 0, \quad (2)$$

$$\nabla_\lambda L = (x + y - 1) = 0, \quad (3)$$

$$\nabla_\mu L = x - 0.5 = 0, \quad \text{if } \mu > 0, \quad (4) \quad \rightarrow x = 0.5$$

Note that if the inequality constraint is *inactive* (i.e. $x - 0.5 < 0$), then μ can be zero. However, if it is *active* (i.e. $x - 0.5 = 0$), then (4) must hold.

2.2.1 MIXED CONSTRAINED OPTIMIZATION

3. Solve the system

Substitute $x = 0.5$ into (3) yields $\quad y = 0.5$

Substitute $y = 0.5$ into (2) yields $\quad \lambda = -1$

Substitute $x = 0.5$ and $\lambda = -1$ into (1) yields $\quad \mu = 0$

4. Verify the KKT conditions

- **Stationarity** Equation (1) and (2) are satisfied with the computed value.
- **Primal feasibility** We have $x + y = 1$ and $g(x, y) = 0.5 - 0.5 = 0$ (thus $x - 0.5 = 0 \leq 0$)
- **Dual feasibility** $\mu = 0 \geq 0$
- **Complementary slackness** $\mu\, g(x, y) = 0 \cdot 0 = 0$

$\therefore$ the candidate optimal point is $(x, y) = (0.5, 0.5)$ with the objective minimum of $f(0.5, 0.5) = 0.5$.

2.2.1 EXERCISES

Use the KKT conditions to find and classify the extrema for the following problems.
(For inequalities, rewrite them in the form $g(x) \leq 0$ if necessary.

1. Minimise $f(x, y) = (x-1)^2 + (y-2)^2$

   subject to $-x - y + 3 \leq 0$ and $-x \leq 0$

2. Maximise $f(x, y) = x^2 y$

   subject to $x + y - 1 = 0$ and $-x \leq 0$

3. Maximise $f(x, y) = \ln(x) + \ln(y)$

   subject to $x + y - 1 = 0, -x \leq 0,$ and $-y \leq 0$

4. Minimise $f(x, y) = e^x + e^y$

   subject to $x + y = 0$ and $x - 1 \leq 0$

5. Minimise $f(x, y, z) = x^2 + y^2 + z^2$

   subject to $x + y + z - 1 = 0$ and $-x \leq 0$