

SUPPLY CHAIN MODELLING AND OPTIMIZATION

TOPIC 7 INVENTORY MODEL WITH DISCOUNT

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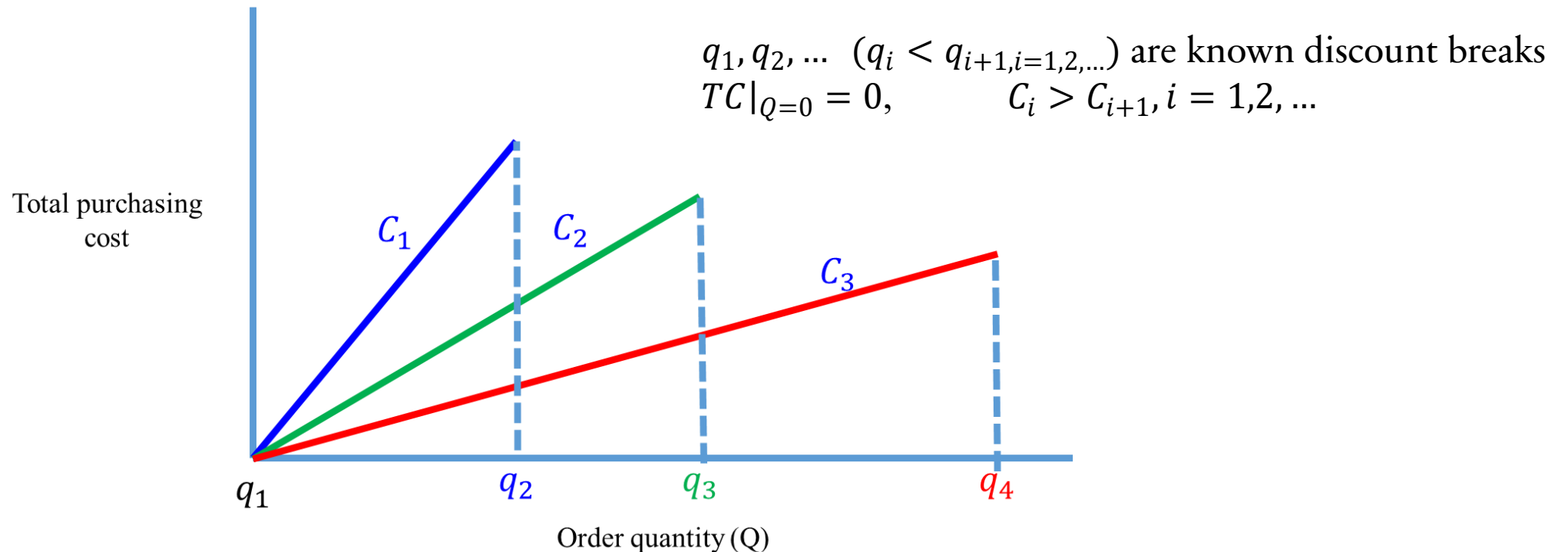
TOPIC 7 INVENTORY MODEL WITH DISCOUNT

1. EOQ Model with Discount
 - 1.1 All units Discount
 - 1.2 Incremental Discount

1 EOQ MODEL WITH DISCOUNT

A **discount** is a reduced price for an item when it is purchased in large quantities. To increase sales, many companies offer quantity discounts to their customers.

A **quantity discount** is simply a decreased unit cost for an item when it is purchased in larger quantities. It is common to have a discount schedule with several discounts for large orders.



1.1 ALL UNITS DISCOUNT

Recall that holding cost also depend on the order quantity, i.e.

$$h_i = I \times C_i, \quad i=1,2,\dots$$

where I is a percentage of the unit purchase cost (C_i) or an internal interest rate covering capital cost and all types of warehousing cost.

Then, the total cost per period when $C = C_i$ is

$$TC_i(Q) = \mathbf{C_i D} + \frac{DA}{Q} + h_i \frac{Q}{2}, \quad i = 1, 2, \dots$$

Note. To avoid confusion, we usually use **D** to represent **annual demand**, and d for demand rate of any unit.

1.1 ALL UNITS DISCOUNT

$$TC_i(Q) = C_i D + \frac{DA}{Q} + h_i \frac{Q}{2}, \quad i = 1, 2, \dots$$

For each C_i , the best order quantity Q_i^* can be established in the following way:

- ✓ $Q_i^* < q_i$: **infeasible**, as we have to order at least b_i units to qualify for the price C_i given this price, it is **best to order b_i units**
- ✓ $q_i < Q_i^* < q_{i+1}$: **feasible**, the optimal decision is Q_i^*
- ✓ $Q_i^* \geq q_{i+1}$: **inefficient**, as we would qualify for an even lower price with this order quantity given this price, we want to order **at most $q_{i+1} - 1$ units**

1.1 ALL UNITS DISCOUNT

When and How much

The major trade-off when considering quantity discounts is between the **reduced item cost** and the **increased carrying cost**.

Table 1. Quantity Discount Schedule (QDS)

Discount Number	Discount Quantity	Discount	Discount Cost
1	0 to 999	0%	\$5.00
2	1000 to 1,999	4%	\$4.80
3	2,000 and over	5%	\$4.75

Buy more – get a discount – more items in the inventory

1.1 ALL UNITS DISCOUNT

What is the right Q^* ?

- Calculate Q^* for each of the unit holding cost

$$Q^* = \sqrt{\frac{2DA}{h_i}} \quad h_i = I \cdot C_i$$

As a result, the value of Q^* will be different for each discounted price.

- For any discount level, if the Q^* value is too low to qualify for the discount, adjust Q^* upward to the lowest quantity that qualifies for the discount.
- For example, if Q^* for discount 2 in the QDS Table turns out to be 500 units (**green TC**), adjust this value up to 1,000 units to get 4% discount (**yellow TC**).

$$TC_i(Q) = C_i D + \left(\frac{D}{Q}\right) A + \left(\frac{Q}{2}\right) I \cdot C_i, \quad q_i \leq Q \leq q_{i+1}$$

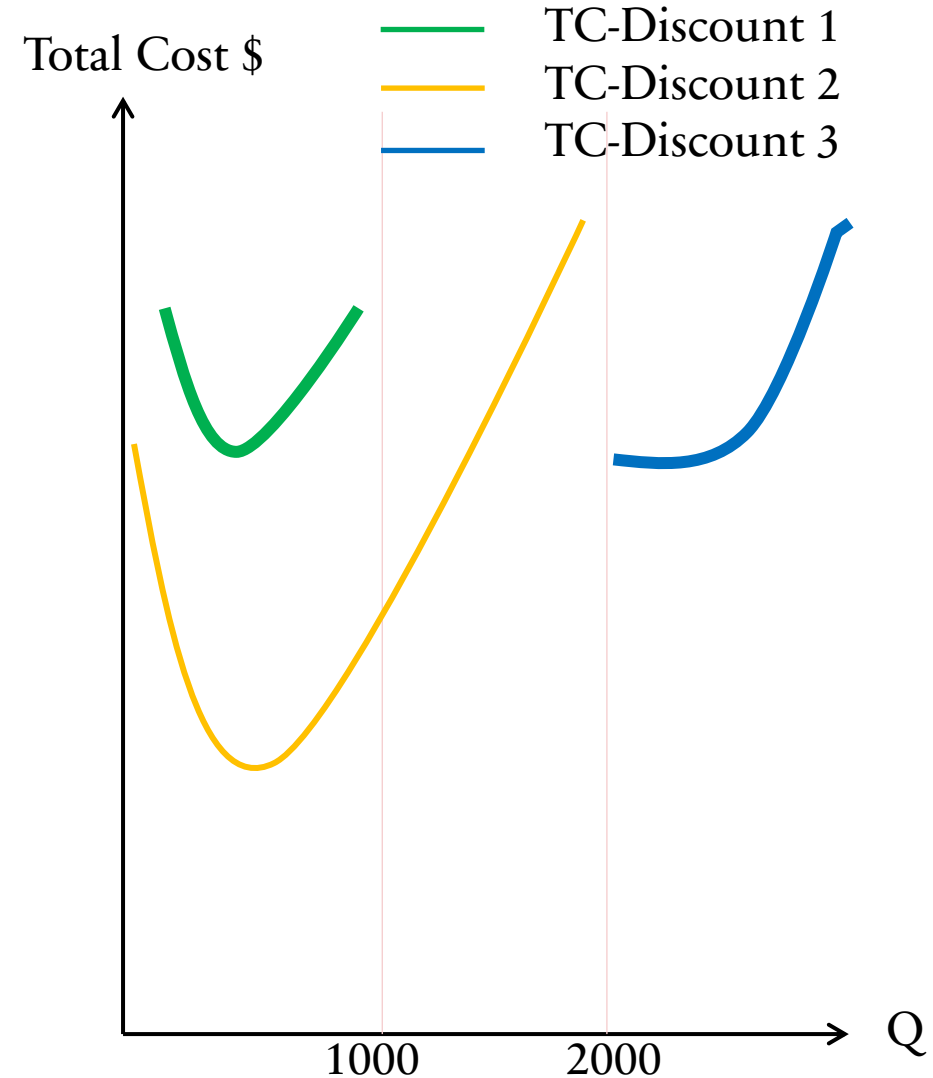


Figure 1. Total Cost according to the QDS Table 1

1.1 ALL UNITS DISCOUNT

To sum up, Q^* can be found by the following procedure:

1. Compute for each i the EOQ: Q_i^*
2. Determine for each i the best admissible order quantity \hat{Q}_i as

$$\hat{Q}_i = \begin{cases} b_i & \text{if } Q_i^* < q_i \\ Q_i^* & \text{if } q_i \leq Q_i^* \leq q_{i+1} \\ q_{i+1} - 1 & \text{if } Q_i^* > q_{i+1} \end{cases}$$

3. Obtain the optimal solution Q^* as

$$Q^* = \hat{Q}_i^* \Big|_{TC(\hat{Q}_i^*)}$$

$$\text{where } TC(\hat{Q}_i^*) = \min_{i=1,2,..} \{TC(\hat{Q}_i)\}$$

1.1 ALL UNITS DISCOUNT

Example 1. Given the following quantity discount schedule table, determine the number of discount level m , discount break q_i and the corresponding cost C_i .

Quantity Purchased	Per-Unit Price
1-100	\$5.00
101-250	\$4.50
251 and higher	\$4.00

Let m be the number of discount possibilities.

From the Table, there are three discount levels, so $m = 3$.

From Table, $q_1 = 0$, $q_2 = 101$, $q_3 = 251$,

$C_1 = \$5$, $C_2 = \$4.50$, $C_3 = \$4$,

1.1 ALL UNITS DISCOUNT – ALGORITHM

Step 1: Set $j = m$, and Compute $Q_m^* = \sqrt{\frac{2DA}{IC_m}}$

Step 2: Is $Q_m^* \geq q_m$?

- If yes, Q_m^* is the optimal order quantity and we are done.
- If not, the minimum cost occurs at $Q = q_m$ owing to the convexity of the cost function.
- Compute the cost TC_{\min} corresponding to $Q = q_m$.

Let $Q_{\min} = q_m$ and go to Step 3.

Step 3: Set $j = j-1$. Compute the optimal EOQ for the j th cost curve: $Q_j^* = \sqrt{\frac{2DA}{IC_j}}$

Step 4: Is Q_j^* in $[q_j, q_{j+1})$?

- If yes, compute $TC(Q_j^*)$ and compare with TC_{\min} .
- If $TC(Q_j^*) < TC_{\min}$, Q_j^* is the optimal order quantity.
- Otherwise, Q_{\min} is the optimal order quantity.

1.1 ALL UNITS DISCOUNT

Example 2. An office supplies store sees a uniform demand rate of 10 boxes of pencils per week. Each box costs \$5. If the fixed cost of placing an order is \$10 and the holding cost rate is 20% per year. Suppose that the retailer gets an all-units discount of 5% per box of pencils if he purchases at least 110 pencil boxes in a single order. The deal is further sweetened if the retailer purchases at least 150 boxes in which case he gets a 10% discount. Determine the optimal order quantity using the EOQ model. Should the retailer change the order quantity?

$$d = 10 \text{ per week}$$

$$C = \$5 \text{ per box}$$

$$A = \$10 \text{ per order}$$

$$I = 20\% \text{ per year}$$

$$q_1 = 0, q_2 = 110, q_3 = 150$$

$$m = 3$$

1.1 ALL UNITS DISCOUNT

Iteration 1: Initialize $j = 3$.

Step 1: We compute $Q_3^* = \sqrt{\frac{2(520)(10)}{(0.2)(4.50)}} = 107.5$.

Step 2: Since Q_3^* is not greater than $q_3 = 150$, the minimum cost occurs at $Q = 150$.

This cost is equal to

$$TC_{min} = (4.50)(520) + \frac{(10)(520)}{150} + \frac{(0.2)(4.50)(150)}{2} = \$2442.17.$$

Step 3: Now, we set $j = 2$. We compute $Q_2^* = \sqrt{\frac{2(520)(10)}{(0.2)(4.75)}} = 104.63$.

Step 4: Once again, Q_2^* is not feasible since it does not lie in the interval $[110, 150)$.

The minimum feasible cost thus occurs at $Q = q_2 = 110$ and is equal to

$$TC(q_2) = (4.75)(520) + \frac{(10)(520)}{110} + \frac{(0.2)(4.75)(110)}{2} = \$2569.52, \quad > TC_{min}$$

1.1 ALL UNITS DISCOUNT

Iteration 2:

Step 3: Now, we set $j = 1$. Recall that the value of Q_1^* is equal to 101.98.

Step 4: Q_1^* is feasible since it lies in the interval $[0, 110)$. The corresponding cost is equal to \$2701.98, which is also higher than TC_{min} .

- Thus, the optimal solution is to order 150 units and the corresponding average annual cost of purchasing and managing inventory is equal to \$2442.17.
- Observe that the algorithm stops as soon as we find a discount for which Q_j^* is feasible.

1.2 INCREMENTAL QUANTITY DISCOUNTS

In the incremental quantity discount, as the quantity per order increases, the unit purchasing **cost declines incrementally on additional units purchased** as opposed to on all the units purchased.

- Let $q_1 = 0, q_2, q_3, \dots, q_j, q_{j+1}, \dots, q_m$ be the order quantities at which the unit purchasing cost changes. The number of discount levels is m .

Assumed that the units are discrete, and the units are infinitely divisible and the purchasing quantity is any real value.

- The unit purchasing cost is the same for all values of Q in $[q_j, q_{j+1})$, and we denote this cost by C_j .
By definition, $C_1 > C_2 > \dots > C_j > C_{j+1} > \dots > C_m$,

1.2 INCREMENTAL QUANTITY DISCOUNTS

Example 3. Let's say a supplier offers the following incremental discount structure:

Quantity Range (units)	Price Per Unit	
1-99	\$10	C_1
100-499	\$9	C_2
500+	\$8	C_3

Purchasing 50 units $50(\$10) = \500

Purchasing 150 units $99(\$10) + 51(\$9) = \$1,449$

Purchasing 600 units $99(\$10) + 400(\$9) + 101(\$8) = \$5,398$

$$m = 3, q_1 = 0, q_2 = 100, q_3 = 500$$

1.2 INCREMENTAL QUANTITY DISCOUNTS

Example 4. The unit purchasing price is \$1.00 for every unit up to 200 units. If the order quantity is between 201 and 500, the unit purchasing price drops to \$0.98 but only for units numbered 201 through 500.

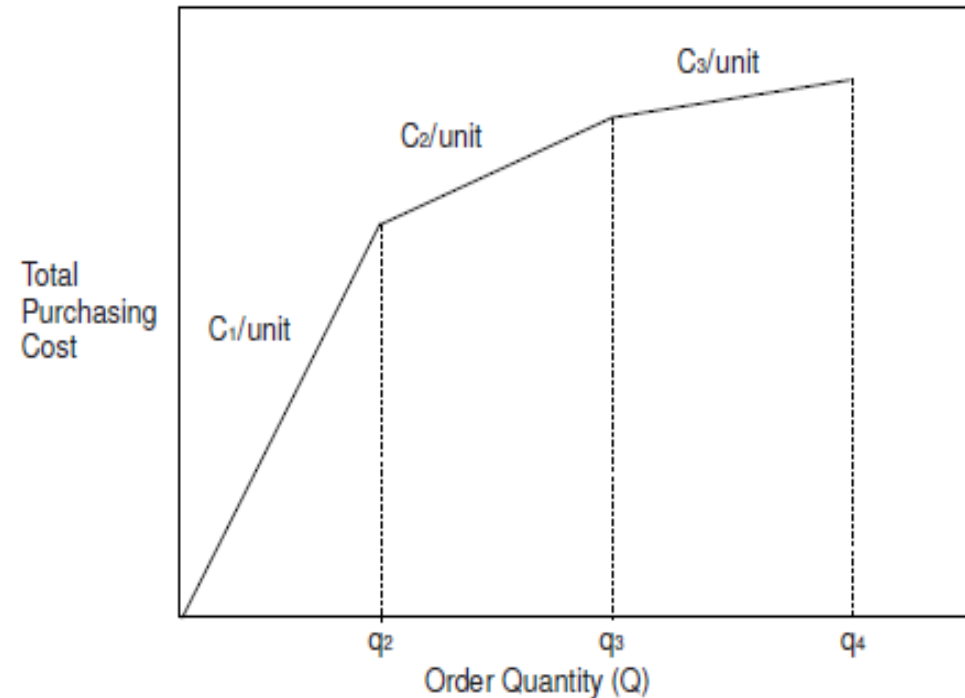
The total purchasing cost of 300 units will be

$$(200)(1) + (300 - 200)(0.98) = \$298.$$

Similarly, if at least 501 units are purchased, the unit purchasing price decreases to \$0.95 for units numbered 501, 502, etc.

$$m = 3, q_1 = 0, q_2 = 201, q_3 = 501$$

$$C_1 = \$1, C_2 = \$0.98, C_3 = \$0.95$$



1.2 INCREMENTAL QUANTITY DISCOUNTS

- Let Q be order quantity in the j th discount interval, that is, Q lies between q_j , and q_{j+1} .
The purchasing cost for Q in this interval is equal to

$$C(Q) = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \dots \\ + C_{j-1}(q_j - q_{j-1}) + C_j(Q - q_j).$$

Note that only the last term on the right-hand side of the above expression depends on Q .

- Let R_j be used to denote the sum of the terms that are independent of Q . That is,

$$R_j = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \dots + C_{j-1}(q_j - q_{j-1}) \quad \text{with } R_1 = 0.$$

Therefore,

$$C(Q) = R_j + C_j(Q - q_j).$$

- Now the average unit purchasing cost for Q units is equal to $\frac{C(Q)}{Q}$, which is equal to

$$\frac{C(Q)}{Q} = \frac{R_j}{Q} + C_j - C_j \frac{q_j}{Q}$$

1.2 INCREMENTAL QUANTITY DISCOUNTS

- The average annual purchasing cost when each order consists of Q units is equal to $\frac{C(Q)}{Q} D$.
- The fixed order cost and holding cost terms remain similar to those in the basic EOQ model except that the average annual holding cost per unit is now equal to $I \frac{C(Q)}{Q}$.
- The average annual cost of managing inventory is thus equal to

$$TC(Q) = \frac{C(Q)}{Q} D + A \frac{D}{Q} + \frac{I \frac{C(Q)}{Q} Q}{2} = \left(\frac{R_j}{Q} + C_j - C_j \frac{q_j}{Q} \right) D + A \frac{D}{Q} + \frac{I(R_j + C_j(Q - q_j))}{2}$$

Realigning terms yields

$$TC(Q) = C_j D + (R_j - C_j q_j + A) \frac{D}{Q} + \frac{I C_j Q}{2} + \frac{I(R_j - C_j q_j)}{2}$$

which is valid for $q_j \leq Q < q_{j+1}$.

1.2 INCREMENTAL DISCOUNTS - ALGORITHM

Step 1: Compute the order quantity that minimizes $\mathbf{TC}_j(Q)$ for each j , which is denoted by Q_j^* and is obtained by setting

$$\frac{d\mathbf{TC}_j(Q)}{dQ} = -(R_j - C_j q_j + A) \frac{D}{Q^2} + \frac{IC_j}{2} = 0$$

$$\rightarrow Q_j^* = \sqrt{\frac{2(R_j - C_j q_j + A)D}{IC_j}}. \quad \text{This step gives us a total of } m \text{ possible order quantities.}$$

Step 2: In this step we check which one of these potential values for Q^* is feasible, that is,

$$q_j \leq Q_j^* < q_{j+1}.$$

Disregard the ones that do not satisfy this inequality.

Step 3: Calculate the cost $\mathbf{TC}_j(Q_j^*)$ corresponding to each remaining Q_j^* . The order quantity Q_j^* that produces the least cost is the optimal order quantity.

1.2 INCREMENTAL DISCOUNTS - ALGORITHM

Example 2 (Extended). Recall the office supplies store example, determine the optimal order quantity assuming the incremental discounts

$$D = 10(52) = 520 \text{ per year} \quad C_1 = \$5, C_2 = 5(0.95) = \$4.75, C_3 = 5(0.9) = \$4.5$$

$$d = 10 \text{ per week}$$

$$C = \$5 \text{ per box}$$

$$A = \$10 \text{ per order}$$

$$I = 20\% \text{ per year}$$

$$q_1 = 0, q_2 = 110, q_3 = 150$$

$$m = 3$$

Step 1: $R_1 = 0$, $R_2 = C_1(q_2 - q_1) = (5)(110 - 0) = 550$, and $R_3 = C_1(q_2 - q_1) + C_2(q_3 - q_2) = R_2 + (4.75)(150 - 110) = 740$. We compute

$$Q_1^* = \sqrt{\frac{2(0 - 0 + 10)(520)}{(0.2)(5)}} = 101.98,$$

$$Q_2^* = \sqrt{\frac{2(550 - (4.75)(110) + 10)(520)}{(0.2)(4.75)}} = 202.61,$$

$Q_2^* \notin [110, 150]$

$$Q_3^* = \sqrt{\frac{2(740 - (4.5)(150) + 10)(520)}{(0.2)(4.5)}} = 294.39.$$

1.2 INCREMENTAL DISCOUNTS - ALGORITHM

Step 2: We disregard Q_2^* since it does not lie in the interval $[110, 150)$. Q_1^* and Q_3^* are feasible.

Step 3: We now compute the costs corresponding to Q_1^* and Q_3^* :

$$\begin{aligned} \text{TC}(Q_1^*) &= (5)(520) + (0 - 0 + 10) \frac{520}{101.98} + \frac{(0.2)(5)(101.98)}{2} + \frac{(0.2)(0 - 0)}{2} \\ &= 2701.98, \end{aligned}$$

$$\begin{aligned} \text{TC}(Q_3^*) &= (4.5)(520) + (740 - (4.5)(150) + 10) \frac{520}{294.39} + \frac{(0.2)(4.5)(294.39)}{2} \\ &\quad + \frac{(0.2)(740 - (4.5)(150))}{2} \\ &= 2611.45. \end{aligned}$$

Thus, the optimal solution is to order 294.39 units and the corresponding average annual cost of purchasing and managing inventory is equal to \$2611.45.

1.2 INCREMENTAL DISCOUNTS - ALGORITHM

Step 2: We disregard Q_2^* since it does not lie in the interval $[110, 150)$. Q_1^* and Q_3^* are feasible.

Step 3: We now compute the costs corresponding to Q_1^* and Q_3^* :

$$\begin{aligned} \text{TC}(Q_1^*) &= (5)(520) + (0 - 0 + 10) \frac{520}{101.98} + \frac{(0.2)(5)(101.98)}{2} + \frac{(0.2)(0 - 0)}{2} \\ &= 2701.98, \end{aligned}$$

$$\begin{aligned} \text{TC}(Q_3^*) &= (4.5)(520) + (740 - (4.5)(150) + 10) \frac{520}{294.39} + \frac{(0.2)(4.5)(294.39)}{2} \\ &\quad + \frac{(0.2)(740 - (4.5)(150))}{2} \\ &= 2611.45. \end{aligned}$$

Thus, the optimal solution is to order 294.39 units and the corresponding average annual cost of purchasing and managing inventory is equal to \$2611.45.